

Stable accelerating universe with no hair

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After reviewing the main characteristics of the spacetime of accelerating universes driven by a quintessence scalar field with a constant equation of state ω , we investigate in this paper the classical stability of such spaces to cosmological perturbations, particularizing in the case of a closed geometry and equation of state $\omega = -\frac{2}{3}$. We conclude that this space is classically stable and conjecture that accelerating universes driven by quintessential fields have "no hair."

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I. INTRODUCTION

Knowing the fate of the universe has always been the main goal of cosmology. The speculation on what the future of the universe might be has now become even more complicated. Following recent observations in supernovas type Ia [1], a real revolution has stirred cosmology, leading to a plethora of new views, concepts and finally to an emerging entirely new cosmological scenario which some like to call the *new standard cosmology* [2]. Critically pivotal in such a scenario is the already overwhelmingly accepted view that the expansion of the present universe is accelerating [3]. Whether this acceleration will continue forever or will stop to recover decelerating expansion is still not known. The key ingredient that provides the repulsive force required by an accelerating universe is cosmic dark energy, which, though not directly detectable, should make up nearly 70% of the total energy content in the universe. So far, two main candidates have been considered for dark energy: a positive cosmological constant [4] and the so-called quintessence field, a slowly varying scalar field with negative pressure which only recently (in cosmic time) has started to dominate over all forms of matter [5,6]. Several compelling arguments have been advanced [5–7] in favor of cosmic quintessence with respect to the cosmological constant.

It has recently been argued, however [8–10], that if the accelerating expansion continues eternally, then a future event horizon will inexorably form and this would represent the demise of any mathematically consistent formulation of quantum gravity and string theory, both in the case in which the universe is now dominated by a positive cosmological constant or by a cosmic quintessential field with a constant [5] or time-dependent [6] equation of state. This may be one of the greatest challenges ever posed to theoretical physics. The formation of a future event horizon, which would make it impossible to construct any consistent S matrix for string theory, is an unavoidable consequence of the causal future development of any initial surface in the accelerating universe if quantum coherence is to be preserved. Nevertheless, there exist solutions to the static Einstein equations for an eternally accelerating universe endowed with a quintessence scalar field with a constant equation of state which do not show any event horizon [11]. This is made possible because such solutions possess a Kleinian signature that can lead to

the allowance of world lines traveling backward in time. Relative to the case of a positive cosmological constant, this may become one of the strongest arguments in favor of quintessence.

Along the evolution of the universe from its quantum era, tracking models can actually be represented [12] as a succession of $\omega = p/\rho = \text{const}$ domains, ranging from $\omega = +1$ to $\omega = -1$, separated by abrupt jumps and followed by the present accelerating expansion with $-\frac{1}{3} > \omega > -1$, which evolves from an attractor solution [6,12,13] and may or may not be characterized by a constant ω . However, it seems to be a good enough approximation to also represent the present period of dark-energy dominated evolution by a quintessence model with $\omega = \text{const}$. Therefore, investigating the spacetime of eternally accelerating universes whose expansion is driven by a quintessence scalar field with a constant equation of state and does not show a future event horizon appears to be of interest. This will be done in the present paper, specializing in the case $\omega = -\frac{2}{3}$. We shall also deal with the issue of the stability of this eternally accelerating universe by considering the cosmological Lifshitz-Khalatnikov perturbations [14] on it. We check that the considered spacetime is stable and we advance the conjecture that an eternally accelerating closed universe has no hair.

The paper can be outlined as follows. In Sec. II, we review the solutions to the Einstein equations corresponding to a quintessence scalar field with constant ω , minimally coupled to Hilbert-Einstein gravity, both in the static and cosmological cases, for the whole range of state equations that covers all possible universes with accelerating expansion. We particularize in the case $\omega = -\frac{2}{3}$ whose Friedmann-Robertson-Walker (FRW) metric is considered in some detail for the different geometries of the universe. The Lifshitz-Khalatnikov cosmic perturbations of this spacetime are studied in detail in Sec. III and it is checked that the spacetime is stable for them. We also advance the conjecture that an eternally accelerating closed universe has no hair, discussing it by comparing with the purely de Sitter space. Finally, we conclude and add some further remarks in Sec. IV.

II. THE SPACETIME OF AN ACCELERATING UNIVERSE

The spacetime of the accelerating Friedmann-Robertson-Walker (FRW) universe endowed with a quintessence scalar

field with constant state equation parameter $-\frac{1}{3} > \omega > -1$ corresponds to maximally symmetric spaces with negative spacetime curvature which are solutions of the Einstein equations. Since the conservation law for a generic quintessence scalar field should be taken to be $\alpha\rho = a(t)^{-3(1+\omega)}$ [where α is an arbitrary integration constant, ρ is the energy density of the quintessence field, and $a(t)$ is the time-dependent scale factor], the Friedmann equations for the quintessential accelerating universe are [15]

$$a^{2+3\omega}\ddot{a} + \frac{4\pi G(1+3\omega)}{3\alpha} = 0, \quad (2.1)$$

$$\frac{\dot{a}^2}{a^2} \equiv H^2 = \frac{8\pi G}{3\alpha} a^{-3(1+\omega)} \pm \frac{1}{R^2 a^2}, \quad (2.2)$$

in which H is the Hubble constant, R^{-2} is the spatial curvature constant, and the overhead dot denotes differentiation with respect to time t . Note that if we set $\omega = -1$, the first term of the right-hand side in Eq. (2.2) becomes a constant, and we obtain then a solution for the scale factor that describes just de Sitter space. Here we shall restrict ourselves to solve Eqs. (2.1) and (2.2) in two particularly interesting cases. When we set $\omega = -\frac{1}{3}$, i.e., at the onset of the accelerating regime, we obtain the solution [16]

$$a(t) = \sqrt{\frac{8\pi G}{3\alpha} - \frac{1}{R^2}} t + K_0, \quad (2.3)$$

where K_0 is an integration constant. Whereas in the spatially closed case with $R^{-2} = 8\pi G/(3\alpha)$ the scale factor (2.3) reduces to a simple constant that describes an Einstein static universe, in the spatially flat and open cases, or when $R^{-2} < 8\pi G/(3\alpha)$ for closed geometry, the universe will expand in just the uniform way.

For a constant equation of state with $\omega = -\frac{2}{3}$, i.e., at the typically most interesting situation in which the universe expands in an accelerating fashion quite adjustable to what has been observed in recent supernova experiments [1,3], the scale factor solution to the Friedmann equations (2.1) and (2.2) can generally be written

$$a(t) = \frac{2\pi G t^2}{3\alpha} + Kt + \frac{3\alpha(K^2 + R^{-2})}{8\pi G}. \quad (2.4)$$

If, without any loss of generality, we set the integration constant $K=0$ (in what follows, it will be seen that when we reexpress the scale factor in terms of the compactified time η , the resulting solution does not explicitly depend on K), then this solution described the FRW spacetime of half a Lorentzian wormhole [17], from a throat at $t=0$, where $a = a_0 = 3\alpha/(8\pi G R^2)$, to the asymptotic region at $t = +\infty$. We furthermore notice that if for $K=0$ we allowed time t also to take on negative values from $t=0$ down to $t = -\infty$, then a complete wormhole would be obtained. Let us next assume for a moment that the two asymptotic regions at $t = \pm\infty$ are identified to each other so that [18] any world lines for test particles or light signals approaching the infinity

surface at $t = +\infty$ from $t=0$ would find themselves emerging from the infinity surface at $t = -\infty$, toward $t=0$, and any of such lines approaching the infinity surface at $t = -\infty$ also from $t=0$ would emerge from the infinity surface at $t = +\infty$, again toward $t=0$. None of these world lines had then reached any of the infinities. Thus, if the universe can be described as a compactified complete wormhole the way we have just described, then light signals traveling backward in time existed and could connect the whole spacetime in such a way that no event horizon would form up in the future [11]. This by no means implies the existence in the accelerating universe of any nonchronal regions containing closed time-like curves because for these curves to occur in our spacetime it would be necessary that the two asymptotic regions at $t = \pm\infty$ were also set into motion relative to one another [18], which is not possible for the universe.

In what follows, we shall interpret the throat of the wormhole at $t=0$ as the latest hypersurface of the universe immediately before the onset of the accelerating regime. This will make the value of the constant $3\alpha/8\pi G$ very large. The three possible geometries associated with the FRW metric that correspond to solution (2.4) with $K \neq 0$ lead to the following expressions for the scale factor in terms of the conformal time $\eta = \int dt/a(t)$.

(i) Spatially closed $R^{-2} > 0$:

$$a(\eta) = \frac{3\alpha}{8\pi G R^2 \cos^2(\eta/(2R))}, \quad (2.5)$$

with

$$\eta = 2R \arctan \left[R \left(K + \frac{4\pi G t}{3\alpha} \right) \right]. \quad (2.6)$$

(ii) Spatially flat $R^{-2} = 0$:

$$a(\eta) = \frac{3\alpha}{2\pi G \eta^2}, \quad (2.7)$$

with

$$\eta = - \frac{2}{K + \frac{4\pi G t}{3\alpha}}. \quad (2.8)$$

(iii) Spatially open $R^{-2} < 0$:

$$a(\eta) = - \frac{3\alpha}{8\pi G |R|^2 \cosh^2(\eta/(2|R|))}, \quad (2.9)$$

with

$$\eta = -2|R| \operatorname{arctanh} \left[R \left(K + \frac{4\pi G t}{3\alpha} \right) \right]. \quad (2.10)$$

Note that in all three cases the scale factor does not explicitly depend on the integration constant K .

Although particular values of the parameter ω such as $\omega = -\frac{1}{3}$ and $\omega = -\frac{2}{3}$ would strictly make the usual sense only for homogeneous and isotropic FRW spacetimes, if we keep an equation of state $p = \omega\rho$ also in the case of spacetimes with static, spherically symmetric coordinates, one can also obtain the static metrics which correspond to the above particular cases for a given, fixed relation between the energy density and the metric components, much in the same way as the static metric for de Sitter space can be derived from the Einstein equations for static, spherically symmetric coordinates and an equation of state $p = \omega\rho$, whenever we set $\omega = -1$. That static de Sitter metric can directly be related with a corresponding FRW metric for de Sitter space obtained from Eqs. (2.1) and (2.2) also for $\omega = -1$ by means of embeddings in a common five-dimensional hyperboloid (see Refs. [20] and [21]). We next consider the generic metric that describes the spacetime in static, spherically symmetric coordinates t, r, θ, ϕ for an equation of state $p = \omega\rho$ and positive spatial curvature, corresponding to an ansatz

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\Omega_2^2,$$

where $d\Omega_2^2$ is the metric on the unit two-sphere, and $A(r)$ and $B(r)$ are metric coefficients depending only on the radial coordinate r . Following Ref. [11], we take as the components of the energy-momentum tensor $T_r^r = T_\theta^\theta = T_\phi^\phi = p$ and $T_0^0 = -\rho$, that is, $T_r^r = T_\theta^\theta = T_\phi^\phi = -\omega T_0^0$, and hence we have the Einstein equations

$$\begin{aligned} 8\pi G\omega\rho &= \frac{1}{B} \left(\frac{1}{r^2} + \frac{A'}{rA} \right) - \frac{1}{r^2} = \frac{1}{4B} \left(\frac{2A''}{A} - \frac{(A')^2}{A^2} - \frac{2B'}{rB} \right. \\ &\quad \left. - \frac{A'B'}{AB} + \frac{2A'}{rA} \right), \\ 8\pi G\rho &= -\frac{1}{B} \left(\frac{1}{r^2} - \frac{B'}{rB} \right) + \frac{1}{r^2}, \end{aligned}$$

in which the prime denotes a derivative with respect to the radial coordinate r . On the other hand, the components of the energy-momentum tensor must satisfy the gravitational equation [11] $T_{i;k}^k = 0$. From this equation, we can now obtain a relation between the metric component $A(r)$ and the energy density ρ , $A(r)' = -2\omega A(r)\rho'/[\rho(1+\omega)]$, which can be immediately integrated to give $\alpha\rho = A(r)^{-(1+\omega)/(2\omega)}$, with α again an arbitrary integration constant. Note that for $\omega = -1$, ρ becomes a simple constant, so that the Einstein equations are straightforwardly solved to produce the well-known static de Sitter metric. In the case $\omega = -\frac{1}{3}$, we can obtain the solution [11,16]

$$A(r) = K_1 - \frac{8\pi G K_2}{3\alpha} \sqrt{1 + \frac{3\alpha K_3}{8\pi G r^2}}, \quad (2.11)$$

$$B(r) = - \left[\left(K_3 + \frac{8\pi G r^2}{3\alpha} \right) A(r) \right]^{-1}, \quad (2.12)$$

where K_1 , K_2 , and K_3 are integration constants which are arbitrary unless by the condition $K_1 K_3 = -1$. This metric shows an event horizon at

$$r = r_h = \left(\frac{3\alpha K_3 K_1^2}{8\pi G K_2^2} - \frac{8\pi G}{3\alpha K_3} \right)^{-1},$$

which marks the transition from a Kleinian-signature metric for $r < r_h$ to a Euclidean (Riemannian) metric for $r > r_h$. For the entire range of ω values in the accelerating interval $-\frac{1}{3} > \omega > -1$, we have the solution [11]

$$A(r) = K(\omega, \alpha) r^{4\omega/(1+\omega)},$$

$$B = \frac{\omega^2 + 6\omega + 1}{(1 * \omega)^2},$$

where

$$\begin{aligned} K(\omega, \alpha) &= \left(\frac{2\pi G(1+\omega)^2 B}{\omega\alpha} \right)^{2\omega/(1+\omega)}, \\ -1 &< \omega < -\frac{1}{3}. \end{aligned}$$

Note that in the considered interval, B reduces to a simple dimensionless constant which is negative definite. Besides being singular at the origin of radial coordinate r , the two most interesting properties of this solution are (i) it does not show any event horizon [so that all world lines should necessarily be always connected to each other in both the static spacetime and, in spite of corresponding to an accelerating FRW spacetime, the cosmological spacetime described by metric (2.4)], and (ii) it has a Kleinian definite signature $(--++)$ [this is consistent with property (i) and with a Lorentzian wormhole interpretation of the FRW metric with scale factor (2.4)]. In this paper, we shall concentrate on the particular value $\omega = -\frac{2}{3}$ which corresponds to a FRW metric that predicts an accelerating universe which conforms well to supernova results. In this particular case, the above solution reduces to

$$A(r) = \left(\frac{3\alpha}{23\pi G r^2} \right)^4, \quad B = -23. \quad (2.13)$$

Most recent results on the large-scale curvature of the universe provided by BOOMERanG [19] and other cosmic microwave background (CMB) anisotropy experiments [20] indicate that the universe is flat with a 95% confidence interval. This leaves a 5% uncertainty which, from the qualitative standpoint, tells us that the geometry of the universe can still be open or closed. On the other hand, along the development of physics we usually find that the apparently most simple situations (circular orbits for planets and atomic electrons, spherical symmetry, etc.) were often not the real case, but just a good approximation. If flatness is taken to represent the simplest possible geometry of the universe, we shall adhere to this empirical tendency by choosing for our accelerating universe with the quintessential equation of state $\omega = -\frac{2}{3}$ a closed geometry, hoping that, as for previous historical

examples, this finally uncovers a richer structure. Moreover, one should expect that the conclusions on the stability of the universe against cosmological perturbations that we are going to obtain in the next section for closed geometry would also be shared by flat geometry. In any event, a detailed consideration of the cases for flat and open geometries is left for future research. It is only for the sake of completeness that we shall finally include the metric solution for any constant ω in the accelerating interval for the flat geometry. In FRW coordinates, it reads

$$ds^2 = -dt^2 + \left(\frac{6\pi G(1+\omega)^2 t^2}{\alpha} \right)^{1/[3(1+\omega)]} \times (dr^2 + r^2 d\Omega_2^2). \quad (2.14)$$

This line element is valid for all $-\frac{1}{3} > \omega > -1$ and is associated with a corresponding static metric for any accelerating value of ω [11] of which the metric coefficients in Eq. (2.13) become just the case for $\omega = -\frac{2}{3}$. Therefore, metric (2.14) and its static counterpart should possess exactly the same properties as were discussed before for metrics (2.4) and (2.13).

III. COSMOLOGICAL PERTURBATIONS

In this section, we shall first briefly review those mathematical aspects of the perturbative technique developed by Lifshitz [14] which will be used later to investigate the stability of our closed accelerating universe. For a closed FRW spacetime, perturbations of the four-metric are introduced by $g_{ab} = g_{ab} + h_{ab}$, with the perturbations satisfying the gauge $h_{00} = h_{0\alpha} = 0, a, b, \dots = 0, 1, \dots, 3$, and $\alpha, \beta, \dots = 1, \dots, 3$. It is work remarking that, although we are working in an isotropic framework with a reference system which is always synchronous, after this choice of gauge it is no longer a comoving system because the perturbations on the spatial components of the four-velocity can be generally nonzero [14,22,23]. Perturbation of the Ricci tensor and, hence, the Einstein equation, energy density, and velocity can then be derived (for details, see Refs. [21,22]) by taking as the most general metric perturbation

$$h_{\alpha\beta} = \lambda(\eta) P_{\alpha\beta} + \mu(\eta) Q_{\alpha\beta} + \sigma(\eta) S_{\alpha\beta} + \nu(\eta) H_{\alpha\beta}, \quad (3.1)$$

where the coefficients λ , μ , σ , and ν depend only on the η time parameter, and $P_{\alpha\beta}$, $Q_{\alpha\beta}$, $S_{\alpha\beta}$, and $H_{\alpha\beta}$ are tensor harmonics which are defined by [22]

$$Q_{\alpha\beta} = \frac{1}{3} \gamma_{\alpha\beta} Q, \quad \nabla^a \nabla_a Q^{(n)} = (1 - n^2) Q^{(n)}, \quad (3.2)$$

$$P_{\alpha\beta} = \frac{\nabla_\alpha \nabla_\beta Q}{l(l+2)} + Q_{\alpha\beta}, \quad S_{\alpha\beta} = \nabla_\alpha S_\beta + \nabla_\beta S_\alpha, \quad (3.3)$$

$$\nabla_a \nabla^a S_b^{(n)} = (2 - n^2) S_b^{(n)}, \quad \nabla^a S_a^{(n)} = 0, \quad (3.4)$$

$$\nabla_a \nabla^a H_{cd}^{(n)} = (3 - n^2) H_{cd}^{(n)}, \quad (3.5)$$

$$\nabla^a H_{ab}^{(n)} = 0, \quad H_a^{(n)a} = 0,$$

where $Q^{(n)}$, $S_a^{(n)}$, and $H_{cd}^{(n)}$ are, respectively, the scalar, vector, and tensor harmonics [21,22], and n is an integer order.

Inserting Eq. (3.1) into the perturbed expressions for the Einstein equations and the energy density and velocity components, and using the above definitions, we finally obtain (a prime denotes differentiation with respect to η)

$$\lambda'' + \frac{2a'\lambda'}{a} - \frac{1}{3}l(l+2)(\lambda + \mu) = 0, \quad (3.6)$$

$$\mu'' + (3C_s^2 + 2) \frac{a'\mu'}{a} + \frac{1}{3}(3C_s^2 + 1) \times [l(l+2) - 3](\lambda + \mu) = 0, \quad (3.7)$$

$$\sigma'' + \frac{2a'\sigma'}{a} = 0, \quad (3.8)$$

$$\nu'' + \frac{2a'\nu'}{a} + l(l+2)\nu = 0, \quad (3.9)$$

and

$$\frac{\delta\rho}{\rho} = \frac{a(\eta=0)^2}{9a^2} \left\{ [l(l+2) - 3](\lambda + \mu) + 3 \frac{a'\mu'}{a} \right\} Q, \quad (3.10)$$

$$\delta v^\alpha = \frac{P^\alpha}{12[1 - (a'/a)^2]} \{ l(l+2)\mu' + [l(l+2) - 3]\lambda' \} \quad (3.11)$$

for scalar harmonics, and

$$\frac{\delta\rho}{\rho} = 0, \quad \delta v^\alpha = [l(l+2) - 3]\sigma' S^\alpha \quad (3.12)$$

for vector harmonics.

A. Scalar harmonics

The differential equations for the η -dependent coefficients λ and μ , representing metric scalar perturbations of a closed FRW geometry for the scale factor (2.5), can be written as

$$\lambda'' + 2 \tan(\eta/2) \lambda' - \frac{1}{3}l(l+2)(\lambda + \mu) = 0, \quad (3.13)$$

$$\mu'' - \frac{1}{3}[l(l+2) - 3](\lambda + \mu) = 0, \quad (3.14)$$

while the perturbations for energy density and velocity components become

$$\frac{\delta\rho}{\rho} = \frac{\cos^4(\eta/2)}{9} \{ [l(l+2)-3](\lambda+\mu) + 3 \tan \eta \mu' \} Q, \quad (3.15)$$

$$\delta v^\alpha = \frac{P^\alpha}{12[1-\tan^2(\eta/2)]} \{ l(l+2)\mu' + [l(l+2)-3]\lambda' \}, \quad (3.16)$$

where we have consistently taken for the speed of sound

$$C_s = \delta p / \delta \rho = \omega = -\frac{2}{3}, \quad (3.17)$$

which corresponds to a quintessence scalar field with constant equation of state $\omega = -\frac{2}{3}$ [12].

The differential equations (3.13) and (3.14) still contain some residual gauge freedom for a complete specification of the choice of coordinates [14,21,22]. Such an unphysical gauge corresponds in the present case to the particular solutions

$$\lambda = -\mu = \text{const}, \quad (3.18)$$

$$\lambda = l(l+2)(\eta + \sin \eta), \quad (3.19)$$

$$\mu = -l(l+2)(\eta + \sin \eta) + 3 \sin \eta.$$

These solutions will be conveniently subtracted from the general solutions to Eqs. (3.13) and (3.14). The latter solutions will be obtained with the help of the auxiliary functions ξ and ζ introduced by the substitutions

$$\lambda + \mu = 3 \sin \eta \int \xi d\eta, \quad (3.20)$$

$$\lambda' - \mu' = 2l(l+2)(1 + \cos \eta) \int \xi d\eta - 3 \cos \eta \int \xi d\eta + \zeta \cos \eta. \quad (3.21)$$

When substitutions (3.20) and (3.21) are introduced in Eqs. (3.13) and (3.14), we obtain the new coupled differential equations

$$\xi' + \cot(\eta/2)\xi + \frac{1}{3}[1 - \frac{1}{2} \sec^2(\eta/2)]\zeta = 0, \quad (3.22)$$

$$\zeta' + [\tan(\eta/2) - \tan \eta]\zeta + \{2l(l+2)(1 + \sec \eta) - 3[1 - \tan \eta \tan(\eta/2)]\}\xi = 0. \quad (3.23)$$

Straightforward manipulations on these equations lead finally to

$$\zeta = -\sec \eta \cot^2(\eta/2) y' \quad (3.24)$$

$$-y'' + \cot(\eta/2)y' + [\frac{2}{3}l(l+2) - \frac{1}{4} + \frac{3}{4} \tan^2(\eta/2)]y = 0, \quad (3.25)$$

where $y = \xi \sin^2(\eta/2)$.

Our task now is to solve Eqs. (3.24) and (3.25). Since obtaining exact solutions to these equations in closed form is very difficult, we shall derive approximate solutions in the extreme cases when $\eta \rightarrow 0$ (i.e., at the beginning of the accelerating phase) and $\eta \rightarrow \pi$ (i.e., toward the asymptotic future of the eternally accelerating expansion). In the first stages of accelerating expansion with $\eta \ll 1$, Eq. (3.25) can be approximated up to second order in η as

$$-\eta y'' + \frac{1}{6}(12 - \eta^2)y' + [\frac{2}{3}l(l+2) - \frac{1}{4}]\eta y = 0. \quad (3.26)$$

At the smallest η the solution can in turn be approximated in terms of the Bessel function J_ν in the form

$$y \approx \eta^{3/2} J_{3/2}[\sqrt{\frac{1}{4} - \frac{2}{3}l(l+2)}\eta]. \quad (3.27)$$

As $\eta \rightarrow 0$, we then have $y \propto \eta^{3/2}$. It follows that for the auxiliary functions $\xi \propto \eta$, $\zeta \propto 4(1 - \eta^2/3)$, and hence, since all residual gauge given by Eqs. (3.10) will vanish as $\eta \rightarrow 0$, one consistently concludes that the metric and dark energy (density and velocity components) perturbations all vanish as one approaches the onset of the accelerating region. Had we chosen for the Bessel function any of the functions H_ν [23], then all the above perturbations would be pure imaginary and divergent as $\eta \rightarrow 0$.

Of greater interest to study the stability of our eternally accelerating universe is to consider the behavior of perturbations in the asymptotic region $\eta \rightarrow \pi$ (i.e., $t \rightarrow \infty$). Thus, we next look at the solutions to Eqs. (3.24) and (3.25) as $\eta \rightarrow \pi$. Let us first introduce the change of time coordinate $x = \tan(\eta/2)$ in Eq. (3.25) which then becomes

$$-(1+x^2)^2 \frac{d^2 y}{dx^2} + \frac{2}{x}(1-x^4) \frac{dy}{dx} + [\frac{8}{3}l(l+2) - 1 + 3x^2]y = 0, \quad (3.28)$$

which for large x and even moderate l admits an approximate solution again expressible in terms of the Bessel function J_ν , i.e.,

$$y \approx \sqrt{\cot(\eta/2)} J_{\sqrt{13}/2}[\sqrt{8l(l+2)-1} \cot(\eta/2)].$$

As $\eta \rightarrow \pi$, this solution reduces to

$$y \propto [\cot(\eta/2)]^{(1+\sqrt{13})/2}, \quad (3.29)$$

or when expressed in terms of the auxiliary functions ξ and ζ ,

$$\xi \approx A(l) \sin^{-2}(\eta/2) [\cot(\eta/2)]^{(1+\sqrt{13})/2}, \quad (3.30)$$

$$\zeta \approx \frac{1}{4} \sec \eta (1 + \sqrt{13}) \xi, \quad (3.31)$$

where $A(l)$ is a finite constant that depends only on l and whose precise value is not of interest in this paper. The use of these expressions in Eqs. (3.20) and (3.21) leads finally to the solutions for the perturbations coefficients

$$\lambda = \frac{A(l)}{3 + \sqrt{13}} [B(l) - 12 \sin^2(\eta/2)] \times [\cot(\eta/2)]^{(5 + \sqrt{13})/2}, \quad (3.32)$$

$$\mu = -\frac{A(l)}{3 + \sqrt{13}} [B(l) + 12 \sin^2(\eta/2)] \times [\cot(\eta/2)]^{(5 + \sqrt{13})/2}, \quad (3.33)$$

where $B(l)$ is a finite constant given by

$$B(l) = \frac{16}{5 + \sqrt{13}} [1 - \sqrt{13}/8 - l(l+2)],$$

and we have subtracted the unphysical gauge associated with the particular solution given by Eqs. (3.19), i.e., $\lambda = 2C_1 l(l+2)(\eta + \sin \eta) + C_2$, $\mu = -2C_1 [l(l+2)(\eta + \sin \eta) - 3 \sin \eta] - C_2$, with C_1 and C_2 two arbitrary integration constants. Using then Eqs. (3.15) and (3.16), we finally derive the perturbation in energy density and velocity components for the quintessence field which are given by

$$\frac{\delta \rho}{\rho} = \frac{2A(l) \cos^4(\eta/2) [\cot(\eta/2)]^{(5 + \sqrt{13})/2}}{3(3 + \sqrt{13}) \csc^2(\eta/2)} \times \left\{ -4[l(l+2) - 3] - 12 \sin \eta \cot(\eta/2) + \frac{5 + \sqrt{13}}{4} [B(l) \csc^2(\eta/2) + 12] \right\} Q, \quad (3.34)$$

$$\delta v^\alpha = \frac{A(l) \cos^2(\eta/2) [\cot(\eta/2)]^{(3 + \sqrt{13})/2}}{(3 + \sqrt{13}) \cos \eta} \left\{ l(l+2) \times \left[\frac{(5 + \sqrt{13})}{4} [B(l) \csc^2(\eta/2) + 12] - 12 \sin [\cot(\eta/2)]^{(2 + \sqrt{13})/2} \right] - [l(l+2) - 3] \times \left[\frac{(5 + \sqrt{13})}{4} [B(l) \csc^2(\eta/2) + 12] + 12 \sin \eta [\cot(\eta/2)]^{(2 + \sqrt{13})/2} \right] \right\} P^\alpha. \quad (3.35)$$

Equations (3.32)–(3.35) vanish as $\eta \rightarrow \pi$. Had we taken any of the two Bessel functions H_ν [23] instead of J_ν for the solution of the differential equation (3.28), then we would have finally obtained the same expressions as Eqs. (3.32)–(3.35), but with the sign for all $\sqrt{13}$ changed. These would again vanish as $\eta \rightarrow \pi$, so the accelerating closed universe resulting from the presence of a quintessence scalar field with constant equation of state $\omega = -\frac{2}{3}$ appears to be stable to scalar Lifshitz-Khalatnikov perturbations.

B. Vector and tensor harmonics

For the case under consideration, perturbations associated with vector harmonics are described by coefficients that satisfy the differential equation

$$\sigma'' + 2 \tan(\eta/2) \sigma' = 0, \quad (3.36)$$

with

$$\frac{\delta \rho}{\rho} = 0, \quad \delta v^\alpha = [l(l+2) - 3] \sigma' S^\alpha. \quad (3.37)$$

The solution to Eq. (3.36) can be given in closed form and reads

$$\sigma = C_0 + C_1 \left(\frac{3}{4} \eta + \sin \eta + \frac{1}{8} \sin(2\eta) \right), \quad (3.38)$$

where C_0 and C_1 are arbitrary integration constants. Now, from Eqs. (3.37) it follows that $\sigma' = 0$, so the constant C_1 should be zero too, and hence $\sigma = C_0$. Therefore, as usual [14,21,22], vector perturbations correspond to unphysical pure gauge and are hence irrelevant also for the eternally accelerating universe we are considering in this paper.

Let us next deal with the quite more interesting study of the gravitational waves associated with the perturbations generated by tensor harmonics. In our case, the coefficients for such perturbations are described by the differential equation

$$\nu'' + 2 \tan(\eta/2) \nu' + l(l+2) \nu = 0. \quad (3.39)$$

Two cases can now be distinguished. If $l=0$, then the solution for $\nu(\eta)$ is formally identical to that for $\sigma(\eta)$ given by Eq. (3.38), but its interpretation is different. It physically represents the time evolution of gravitational waves. We have obtained that, even though the gravitational wave amplitude does not vanish as $\eta \rightarrow \pi$, it does not grow as $t \rightarrow \infty$, at which limit $\nu = C_* \equiv C_0 + 3\pi C_1/4 = \text{const}$. The neat effect of the whole evolution from $\eta=0$ to $\eta=\pi$ on the zero-mode amplitude is an increase from C_0 to C_* . If $l \neq 0$, then by introducing the coordinate change $z = \sin(\eta/2)$, Eq. (3.39) can be written as

$$(1 - z^2) \nu'' + 3z \nu' + m(m+4) \nu = 0, \quad (3.40)$$

in which $m = 2l$. The solution to this differential equation can most easily be expressed in terms of ultraspherical (Gegenbauer) polynomials $C_n^{(3)}$ [23] of odd degree n and reads

$$\nu(\eta) = \cos^{5/2}(\eta/2) C_{2l-1}^{(3)}[\sin(\eta/2)], \quad l > 0. \quad (3.41)$$

We can then show that this solution becomes proportional to $t^{-5/2}$ as one approaches the asymptotic limit $\eta = \pi$. Thus, these gravitational modes (which all start with vanishing amplitude at $\eta=0$) become asymptotically suppressed as one goes to $t = \infty$. We have thereby excluded any unstable growing modes of the gravitational radiation in the accelerating regime driven by a quintessence field with constant equation of state $\omega = -\frac{2}{3}$.

C. No-hair conjecture

Now, from the solution (3.38) for $l=0$ we obtain at large t ,

$$\nu_{\omega=-2/3}^{(0)} \approx C_* + \sqrt{\frac{27\alpha}{128\pi G a}} C_1. \quad (3.42)$$

Thus, even though the value of ν provided by Eq. (3.30) would be expected to be larger than the corresponding value for the zero mode, $\nu_{ds}^{(0)} = C_2 = \text{const}$, for de Sitter space [21], one may always choose the value of the constant C_* , C_1 , and C_2 such that $\nu_{\omega=-2/3}^{(0)}$ becomes smaller than $\nu_{ds}^{(0)}$ at sufficiently later times. For $l \neq 0$, from solution (3.41) at large t , we also obtain

$$\nu_{\omega=-2/3}^{(l)} \approx \frac{(2l+4)!}{5!(2l-1)!} (2a)^{-5/4}. \quad (3.43)$$

The comparison of this with the corresponding de Sitter expression [21] $\nu_{ds}^{(l)} \approx A_1 + A_2 \exp(-3Ht)$ (where H is the Hubble constant and A_1 and A_2 are constants which only depend on l) clearly implies that $\nu_{\omega=-2/3}^{(l)} < \nu_{ds}^{(l)}$ even at moderate values of time t .

Moreover, from the above discussion it follows that any physical effects driven by the gravitational radiation modes (3.42) and (3.43) should in any event be very small, since their physical wavelengths, respectively, increase with $a^{1/6}$ and $a^{5/12}$. Thus physical quantities involving at least two derivatives of the metric are then suppressed asymptotically by powers of the inverse of these wavelengths. These results appear to be implying a cosmic “no-hair” theorem [24] for our accelerating closed universe endowed with a quintessence field with constant equation of state $\omega = -\frac{2}{3}$. According, furthermore, to our discussion above, it would rather be a question of the relative values of the constants involved in the expressions for coefficients ν whether the no-hair de Sitter attractor or our no-hair attractor is the final solution for an accelerating universe. Of course, the model discussed in this paper is a simple one, so that one should extend this discussion to include other models with different spatial geometries and constant or “tracking” equations of state.

IV. CONCLUSIONS

Motivated by the huge impact produced by supernova observations [1], in this paper we have studied some new characteristics of the cosmological and static spacetimes of accelerating universes whose expansion is driven by a slowly varying quintessence scalar field with constant equation of state $-\frac{1}{3} > \omega > -1$. Particularizing at the observationally most favored case $\omega = -\frac{2}{3}$, it was shown that these spacetimes do not possess any future event horizon, a property which is not obviously shared by de Sitter space. The reason that justifies this result is that, even though closed timelike curves are not permitted to exist in these spacetimes, world lines traveling backward in time are allowed to occur on it,

thus connecting any otherwise causally disconnected regions, and hence preventing the formation of future event horizons. This result could have considerable interest for particle physics and quantum gravity because it manifestly avoids the serious, perhaps fatal difficulties for string theory (and actually any quantum field theory that depends on the presence of particles at infinity) posed by the existence of a future event horizon [8–10]. This would be just another reason in favor of preferring quintessence over a positive cosmological constant, since asymptotically de Sitter space inexorably leads to the formation of an event horizon.

The stability of the closed space with accelerating expansion induced by a quintessence field with an equation of state $\omega = -\frac{2}{3}$ to the cosmological Lifshitz-Khalatnikov perturbations [14] on the three-sphere has been also studied in detail. Although there exist more elaborate, covariant methods for dealing with cosmological perturbations [25], we have followed here the original Lifshitz-Khalatnikov treatment because of its greater ability to distinguish among the involved physical effects. It was obtained that, at least for a closed geometry in the case $\omega = -\frac{2}{3}$, the space is stable to both the scalar and tensorial perturbations, and that the damping of small physical effects induced by the resulting gravitational waves allows one to conjecture that—much like it happens in asymptotic de Sitter space—eternally accelerating universes induced by quintessential fields have no hair, thus becoming final attractors along the evolution of the universe. It would be the value taken on by the constants that characterize the amplitude of the gravitational radiation that would finally decide whether the attractor of a de Sitter universe or the attractors of $\omega > -1$ quintessential accelerating universes are going finally to dominate and drive the future cosmological evolution. This is a matter which cannot be decided in the present paper, but that appears to be necessary to avoid the above-mentioned severe conflict between the accelerating universe and string theory (or possibly any competing quantum theory, if there were any). Since our present understanding of any of the theories involved in this conflict is still too rudimentary, it is rather premature to say anything definite about it.

It is expected that the results obtained in this paper for the closed geometry can be applied to flat geometry too. In particular, it appears that the no-hair conjecture is a key ingredient also for the spatially flat case. Finally, the rather intriguing implication that in any eternally accelerating universe driven by quintessence there would be world lines (followed by light signals and possibly some kind of matter) traveling backward in time needs further consideration. After all, if such lines were allowed to exist, then the meaning of cosmological evolution itself would actually require an extensive revision.

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